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Odd Sum Labeling for Complete Bipartite Graph and its Splitting and Subdivision

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ABSTRACT: In this article the discussion has been carried out on odd sum labeling of a complete bipartite graph $K_{2,n}$, splitting graph of $K_{2,n}$, subdivision of $K_{2,n}$, a super subdivision of cycle C_{4n} when each edge of the cycle is replaced by $K_{2,t}$ and an arbitrary super subdivision of path P_n when each edge of the path is replaced by K_{2,m_i} with arbitrary m_i. The aim of this paper is to bring together original research and review articles discussing recent developments in graph labeling. This paper summary is to show the path for new learners to work in the field of odd sum labeling of graphs.

Keywords: Odd sum labeling, odd sum graph, complete bipartite graph, subdivision, super subdivision, arbitrary super subdivision.

I. INTRODUCTION

Most graph labelings trace their origins to labelings presented by Rosa [1]. He identified three types of labelings, which he called $\alpha, \beta -$, and $\rho -$labelings. $\beta -$labeling was renamed as graceful labeling by Solomon Golomb. A detailed survey on graph labeling is updated every year by Gallian [2]. Somasundaram and Ponraj [3] have studied the mean labeling of a path, cycle and complete graph. Mean labeling of a double triangular snake graph and a double quadrilateral snake graph was studied by Gayathri and Gopi [4]. The idea of odd sum labeling was given by Arockiaraj and Mahalakshmi [5] with odd sum labeling of a path, cycle, balloon graph, ladder graph, quadrilateral snake graph, bistar graph and cyclic ladder graph. Arockiaraj *et al.* [6, 7] discussed the odd sum property of some subdivision graphs and graphs obtained by duplicating any edge of some graphs. Gopi [8] investigated odd sum labeling of some tree-related graphs such as the H graph of a path, the twig graph, the graph $P(m,n)$ and the graph (P_m, S_n). Gopi and Iraudaya Mary [9] studied the odd sum property for the slanting ladder graph, the shadow graph of a star graph and bistar graph, the mirror graph of a path and the graph obtained by duplicating a vertex in a path. The concept of odd-even sum labeling was first given by Monika and Murugan [10] with odd-even sum labeling of path, star, bistar and some standard graphs. General results on odd-even sum labeling are presented by Kaneria and Andharia [11]. Andharia and Kaneria [12] defined even sum labeling. Even sum labeling of various graphs are presented by Kaneria and Andharia [13, 14].

Odd sum labeling and odd sum graph is defined in [5] as, "An injective function $f: V(G) \rightarrow \{0, 1, 2, \dots\dots, |E(G)|\}$ is said to be an odd sum labeling if the induced edge labeling f^* defined by $f^*(uv) = f(u) + f(v)$, $\forall\, uv \in E(G)$ is a bijective and $f^*: E(G) \rightarrow \{1, 3, 5, \dots\dots, 2|E(G)| - 1\}$. A graph is said to be an odd sum graph if it admits an odd sum labeling".

This article deals with odd sum labeling of the graph $K_{2,n}$, a splitting graph of $K_{2,n}$, a subdivision of $K_{2,n}$, a supersubdivision of cycle C_{4n} when each edge of the cycle is replaced by $K_{2,t}$ and an arbitrary super subdivision of path P_n when each edge of the path is replaced by K_{2,m_i} with arbitrary m_i.

Definition: For each vertex v of a graph G, take a new vertex v'. Join v' to all vertices of G adjacent to v. The graph $S'(G)$ thus obtained is called the splitting graph of G. Clearly, if a graph G is (p, q) graph then the splitting graph of G is a $(2q, 3q)$ graph [15].

Definition: If each edge of a graph G is broken into two edges by exactly one vertex, then the resultant graph is said to be a subdivision of G and it is denoted by $S(G)$.

Definition: A super subdivision of a graph G, denoted by $SS(G)$ is a graph obtained from G by replacing every edge xy of G with a complete bipartite graph $K_{2,t}$, for some t in such a way that the end vertices x, y of each edge are merged with the two vertices of 2-vertices part of $K_{2,t}$ after removing the edge xy from G [16].

Definition: A super subdivision of a graph G is said to be an arbitrary super subdivision of a graph G if every edge of a graph G is replaced by an arbitrary $K_{2,m}$ where m varies arbitrarily. We shall denote it by $ASS(G)$.

II. MAIN RESULTS

Theorem 1: The graph $K_{2,n}$ $(n \geq 1)$ is an odd sum graph.

Proof: Consider a (p,q)graph G which is a complete bipartite graph $K_{2,n}$, $n \geq 1$ where p is the number of vertices in G and q is the number of edges in G. Let the vertex set of G is $V(G) = \{u_1, u_2\} \cup \{v_i : i = 1,2, \ldots \ldots, n\}$ then the edge set of G is $E(G) = \{u_1 v_i, u_2 v_i : i = 1,2, \ldots \ldots, n\}$ and hence $p = n + 2$ and $q = 2n$.

We define the vertex labeling function $f: V(G) \rightarrow \{0,1,2, \ldots \ldots, q\}$ as

$f(u_1) = 0, \quad f(u_2) = q$

and $f(v_i) = 2i - 1, \quad$ where $i = 1,2, \ldots \ldots, n$.

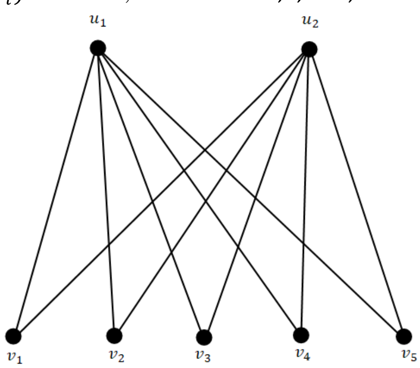

Fig. 1. Ordinary labeling of $K_{2,n}$.

Then, the induced edge labels are

$$f^*(u_1 v_i) = 2i - 1, \quad \forall\, i = 1,2, \ldots \ldots, n;$$

$$f^*(u_2 v_i) = 2n + 2i - 1, \ \forall\, i = 1,2, \ldots \ldots, n.$$

Thus, $f^*(uv) \in \{1,3,5, \ldots \ldots, 2q - 1\}$, $\forall\, uv \in E(G)$. So, $f^*\big(E(G)\big) = \{1, 3, 5, \ldots \ldots, 2q - 1\}$.

Therefore, f is the odd sum labeling of G, and hence, $K_{2,n}$, $n \geq 1$ is an odd sum graph.

Illustration 1:

Odd sum labeling of a complete bipartite graph $K_{2,5}$ is shown in Fig. 2.

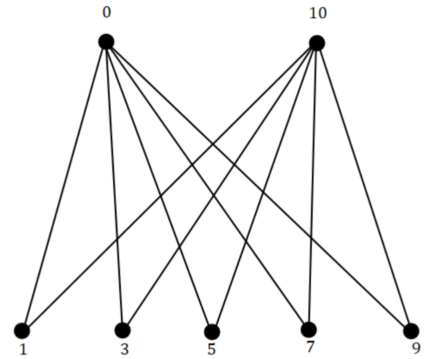

Fig. 2. Odd sum labeling of $K_{2,5}$.

Theorem 2: $S^{'}(K_{2,n})$, the splitting graph of $K_{2,n}$, is an odd sum graph.

Proof: Let v_1, v_2 and u_i, $i = 1,2, \ldots \ldots, n$ be the vertices of a graph $K_{2,n}$. Let $u_i^{'}$ be the added vertices corresponding to each u_i and $v_1^{'}$ and $v_2^{'}$ be the vertices added corresponding to v_1 and v_2 respectively. To obtain $S^{'}(K_{2,n})$, each $u_i^{'}$ is joined to v_1 and v_2 as well each u_i is joined to $v_1^{'}$ and $v_2^{'}$ by an edge as shown in Fig. 3.

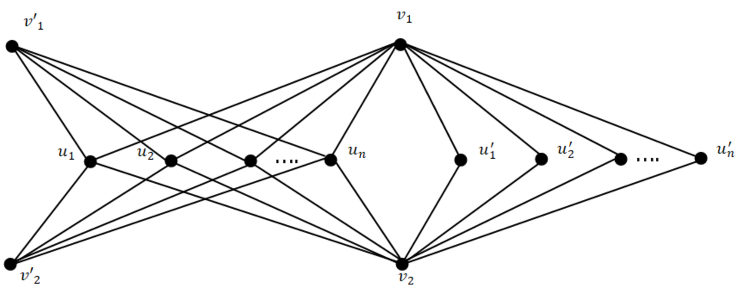

Fig. 3. The splitting graph of $K_{2,n}$.

Clearly, $E\left(S^{'}(K_{2,n})\right) = \{u_i v_1,\ u_i v_2,\ v_1 u_i^{'},\ v_2 u_i^{'},\ u_i v_1^{'},\ u_i v_2^{'} :\ i = 1,2, \ldots \ldots, n\}$.

So, $\left|E\left(S^{'}(K_{2,n})\right)\right| = 6n$ or $q = 6n$.

The vertex labeling of the graph $S^{'}(K_{2,n})$ is defined as

$$f(v_1) = 0;$$
$$f(v_2) = 2n;$$
$$f(u_i) = 4n + (2i - 1);\ \forall\, i = 1,2, \ldots \ldots, n;$$
$$f(v_1^{'}) = 4n;$$
$$f(v_2^{'}) = 6n;$$
$$f(u_i^{'}) = 2i - 1;\ \forall\, i = 1,2, \ldots, n.$$

The induced edge labeling f^* for the graph $S^{'}(K_{2,n})$ is given by

$f^*(uv) = f(u) + f(v),\ \forall\, uv \in E(S^{'}(K_{2,n}))$.

$$\therefore\ f^*(u_i v_1) = 4n + (2i - 1), \quad \forall\, i = 1,2, \ldots \ldots, n$$
$$= \{4n + 1, \quad 4n + 3, \ldots \ldots, \quad 6n - 1\};$$
$$f^*(u_i v_2) = 6n + (2i - 1), \quad \forall\, i = 1,2, \ldots \ldots, n$$
$$= \{6n + 1, \quad 6n + 3, \ldots \ldots, \quad 8n - 1\};$$
$$f^*(v_1 u_i^{'}) = 2i - 1, \quad \forall\, i = 1,2, \ldots \ldots, n$$
$$= \{1, 3, \ldots \ldots, 2n - 1\};$$
$$f^*(v_2 u_i^{'}) = 2n + 2i - 1, \quad \forall\, i = 1,2, \ldots \ldots, n$$
$$= \{2n + 1, \quad 2n + 3, \ldots \ldots, 4n - 1\};$$
$$f^*(u_i v_1^{'}) = 8n + (2i - 1), \quad \forall\, i = 1,2, \ldots \ldots, n$$
$$= \{8n + 1, \quad 8n + 3, \ldots \ldots, 10n - 1\};$$
$$f^*(u_i v_2^{'}) = 10n + (2i - 1), \quad \forall\, i = 1,2, \ldots \ldots, n$$
$$= \{10n + 1, \quad 10n + 3, \ldots \ldots, 12n - 1\}.$$

Thus,

$f^*(uv) \in \{1, 3, 5, \ldots \ldots, 12n - 1\} = \{1, 3, 5, \ldots \ldots, 2q - 1\}$, $\forall\, uv \in E\left(S^{'}(K_{2,n})\right)$.

So, $f^*\left(E\left(S^{'}(K_{2,n})\right)\right) = \{1, 3, 5, \ldots \ldots, 2q - 1\}$.

Therefore, f is the odd sum labeling of $S^{'}(K_{2,n})$, and hence, $S^{'}(K_{2,n})$ is an odd sum graph.

Illustration 2:

The odd sum graph $S^{'}(K_{2,4})$ is shown in Fig. 4.

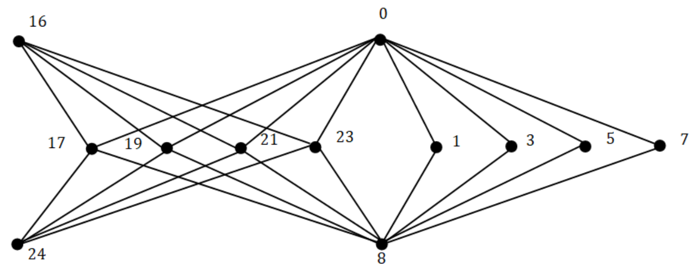

Fig. 4. Odd sum labeling of $S'(K_{2,4})$.

Theorem 3: A subdivision of $K_{2,n} (n \in N)$ admits an odd sum labeling.

Proof:

Let $V(K_{2,n}) = \{u, v\} \cup \{w_i : i = 1,2, \dots \dots, n\}$ and $E(K_{2,n}) = \{uw_i : i = 1,2, \dots \dots, n\} \cup \{vw_i : i = 1,2, \dots \dots, n\}$.

Let u_i be the vertex that divides the edge uw_i for each $1 \leq i \leq n$ and v_i be the vertex that divides the edge vw_i for each $1 \leq i \leq n$. The resultant graph $G = S(K_{2,n})$ is shown in Fig. 5.

Thus, $V(G) = \{u, v\} \cup \{u_i, v_i, w_i : i = 1,2, \dots \dots, n\}$ and $E(G) = \{uu_i, vv_i, u_iw_i, v_iw_i : i = 1,2, \dots \dots, n\}$.

Clearly, $q = |E(G)| = 4n$.

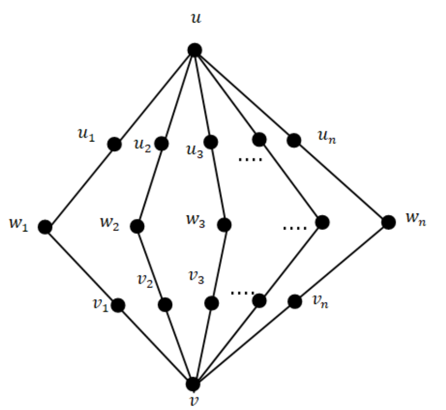

Fig. 5. Subdivision of $K_{2,n}$.

We define a function $f: V(G) \to \{0,1,2, \dots \dots, q\}$ as

$f(u) = 0;\quad f(v) = q;$

$$f(u_i) = 2i - 1, \qquad \forall\, i = 1,2, \dots \dots, n;$$

$$f(v_i) = \frac{q}{2} + 2i - 1, \qquad \forall\, i = 1,2, \dots \dots, n;$$

$$f(w_i) = q + 2 - 4i, \qquad \forall\, i = 1,2, \dots \dots, n.$$

The induced edge labeling function $f^*: E(G) \to \{1,3,5, \dots \dots, 2q - 1\}$ is given by

$f^*(xy) = f(x) + f(y), \forall\, xy \in E(G)$.

The above stated labeling pattern gives rise to odd sum labeling of $S(K_{2,n})$. Hence, a subdivision of $K_{2,n} (n \in N)$ admits an odd sum labeling.

Illustration 3: The subdivision of $K_{2,5}$ with its odd sum labeling is shown in Fig. 6.

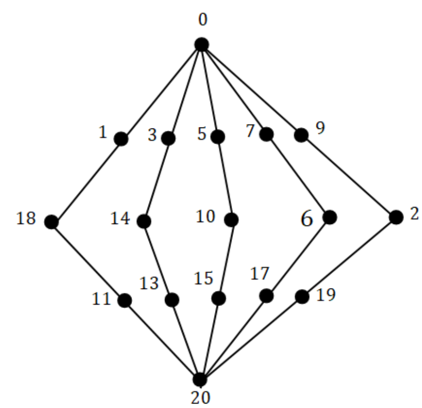

Fig. 6. Odd sum labeling of subdivision of $K_{2,5}$

Theorem 4: A super subdivision of cycle C_{4n} when each edge is replaced by $K_{2,t}$ is an odd sum graph.

Proof: Suppose $u_1, u_2, \dots \dots \dots, u_{4n-1}, u_{4n}$ are the vertices of given cycle C_{4n}.

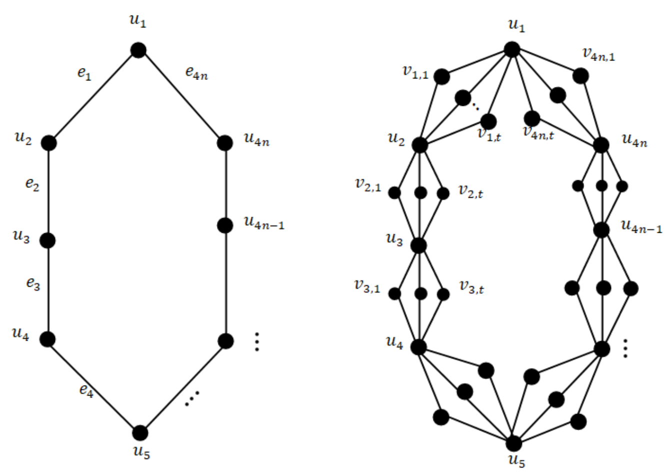

Fig. 7. Cycle C_{4n} and its super subdivision.

Let $e_i = u_i u_{i+1}, \forall\, i = 1,2,\dots\dots,4n-1$ and $e_{4n} = u_{4n}u_1$ be the edges of the cycle C_{4n}. Fig. 7 shows a cycle C_{4n} and its super subdivision $SS(C_{4n})$ which is obtained by replacing each edge $e_i, i = 1,2,\dots\dots,4n$ by a complete bipartite graph $K_{2,t}$ for some positive integer t in such a way that the end vertices of each edge e_i are merged with the two vertices of 2-vertices part of $K_{2,t}$. We take $G = SS(C_{4n})$ where

$V(G) = \{u_i : i = 1,2,\dots\dots,4n\} \cup \{v_{i,j} : i = 1,2,\dots\dots,4n; j = 1,2,\dots\dots,t\}$ and

$$E(G) = \{u_i v_{i,j}, v_{i,j} u_{i+1} : i = 1,2,\dots\dots,4n-1; j = 1,2,\dots\dots,t\} \cup \{u_{4n} v_{4n,j}, v_{4n,j} u_1 : j = 1,2,\dots\dots,t\}.$$

Now, we define a function $f: V(G) \to \{0,1,2,\dots\dots,q\}$ as follows, where $q = |E(G)| = 8nt$.

$$f(u_i) = 2t(i-1), \forall\, i = 1,2,\dots\dots,2n; f(u_i) = 2ti, \forall\, i = 2n+1, 2n+2,\dots\dots,4n;$$

$$f(v_{i,j}) = 2t(i-1) + 2j - 1, \forall\, i = 1,2,\dots\dots,4n;\ \forall\, j = 1,2,\dots\dots,t.$$

The above vertex labeling pattern with the induced edge labeling function $f^*: E(G) \to \{1,3,5,\dots\dots,2q-1\}$ given by $f^*(xy) = f(x) + f(y), \forall\, xy \in E(G)$ implies the odd sum labeling of G. Hence, a super subdivision of C_{4n} when each edge of the cycle is replaced by $K_{2,t}$ is an odd sum graph.

Illustration 4: Fig. 8 shows the odd sum labeling of the super subdivision of cycle C_8 when each edge of C_8 is replaced by $K_{2,3}$.

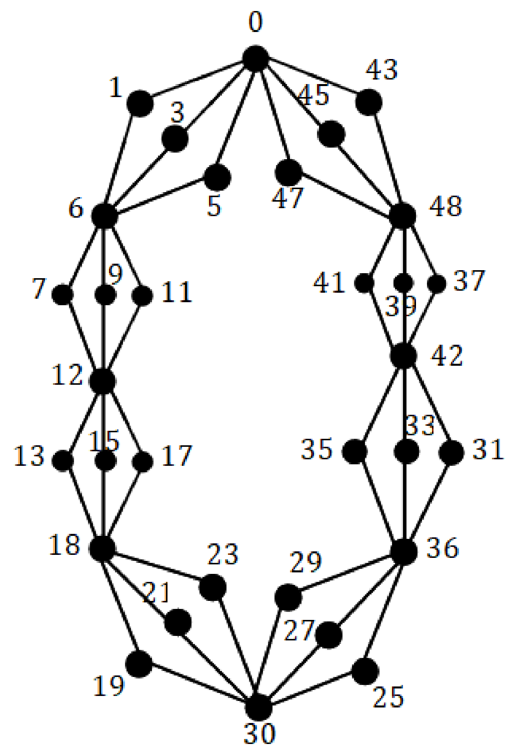

Fig. 8. Odd sum labeling of super subdivision of C_8 when each edge is replaced by $K_{2,3}$.

Theorem 5: An arbitrary super subdivision of path P_n when each edge of the path is replaced by K_{2,m_i} with arbitrary m_i is an odd sum graph.

Proof: Suppose $u_1, u_2, \dots\dots\dots, u_n$ are the vertices of given path P_n. Let $e_i = u_i u_{i+1}, \forall\, i = 1,2,\dots\dots,n-1$ be the edges of P_n. Let G be an arbitrary super subdivision of P_n *i.e.* for $1 \le i \le n-1$, each edge e_i of P_n is replaced by a complete bipartite graph K_{2,m_i} with arbitrary positive integer m_i.

Clearly,

$V(G) = \{u_i : i = 1,2,\dots\dots,n\} \cup \{v_{ij} : i = 1,2,\dots\dots,n-1; j = 1,2,\dots\dots,m_i\}$ and

$E(G) = \{u_i v_{ij}, u_{i+1} v_{ij} : i = 1,2,\dots\dots,n-1;\ j = 1,2,\dots\dots,m_i\}$.

Thus, $q = |E(G)| = 2\sum_{i=1}^{n-1} m_i$.

Now, define a vertex labeling function $f: V(G) \to \{0,1,2,\dots\dots,q\}$ as

$f(u_1) = 0;$

$$f(u_i) = 2\sum_{j=1}^{i-1} m_j, \qquad \forall\, i = 2,3,\dots\dots,n;$$

$$f(v_{ij}) = f(u_i) + (2j-1), \forall\, i = 1,2,\dots\dots,n-1;\ \forall\, j = 1,2,\dots\dots,m_i.$$

The induced edge labeling function $f^*: E(G) \to \{1,3,5,\dots\dots,2q-1\}$ is given by

$f^*(xy) = f(x) + f(y), \forall\, xy \in E(G)$.

The above labeling pattern shows the odd sum labeling of G. Hence, an arbitrary super subdivision of path P_n when each edge of the path is replaced by K_{2,m_i} with arbitrary m_i is an odd sum graph.

Illustration 5: Arbitrary super subdivision of path P_7 with its odd sum labeling is shown in Fig. 10.

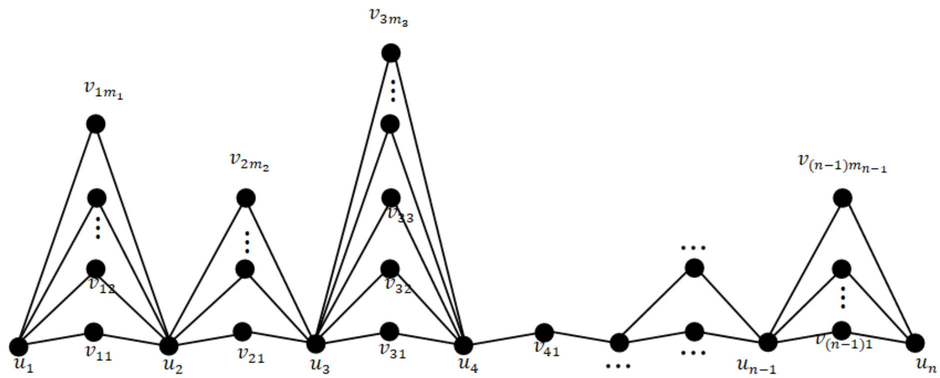

Fig. 9. Arbitrary super subdivision of path P_n.

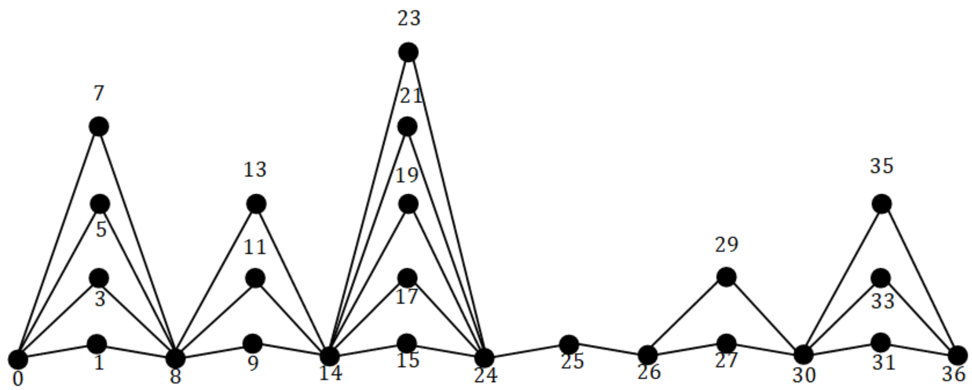

Fig. 10. Odd sum labeling of arbitrary super subdivision of P_7.

III. CONCLUSION

The attempt was made to identify odd sum labeling of $K_{m,n}$, splitting of $K_{m,n}$ and subdivision of $K_{m,n}$, we found that the results are relevant for $m = 2$. We also took initiative to establish the odd sum labeling of a super subdivision of cycle C_{4n} when each edge of the cycle is replaced by $K_{2,t}$ and an arbitrary super subdivision of path P_n when each edge of the path is replaced by K_{2,m_i} with arbitrary m_i.

IV. FUTURE SCOPE

One can attempt to establish odd sum labeling of $K_{m,n}$, splitting of $K_{m,n}$ and subdivision of $K_{m,n}$ for arbitrary values of m. Researchers of this field can find more graphs which follow the property of odd sum labeling. Scope of investigation also holds true in applying odd sum labeling property in context to different graph operations.

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